21-241 Final Exam Solutions
Fall 2024

1. Note that this problem spans two pages. Determine whether the following statements are true or false. If a statement is true, prove it; if it is false, prove it or provide a counterexample.

   (a) Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ be vectors such that $\mathbf{w}$ is not a multiple of $\mathbf{u}$ and $\mathbf{v}$. Then the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent.

   False. Let $\mathbf{u} = \begin{bmatrix}1\\1\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}2\\2\end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix}1\\0\end{bmatrix}$. Then $\mathbf{w}$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$, but $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is not linearly independent since $\mathbf{u}$ and $\mathbf{v}$ are multiples of each other.

   (b) If the row space of an $n \times n$ matrix A is $\mathbb{R}^n$, then A is invertible.

   True. First recall that $\dim C(A^T) = \text{rank}A$, which is the number of pivots of A. Since $C(A^T) = \mathbb{R}^n$, we have that $\text{rank}A = n$ and so A has n pivots. Hence A is invertible.

   (c) Let P be a projection matrix that projects vectors in $\mathbb{R}^5$ onto a $3-$dimensional subspace W of $\mathbb{R}^5$. Then $\det P = 0$.

   True. Since P projects onto W, we have that $W = C(P)$. Hence $C(P)$ is 3-dimensional and so P is a 5×5 matrix of rank 3. Hence P does not have a pivot in every row/column and so it is not invertible. It follows that $\det P = 0$.

   (d) If Q is a matrix with orthonormal columns, then $QQ^T = I$.

   False. Let $Q = \begin{bmatrix}1\\0\end{bmatrix}$. Then Q has orthonormal columns, but $QQ^T = \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$.

   (e) $W = \left\{ \begin{bmatrix}x\\y\\1\end{bmatrix} \in \mathbb{R}^3 \,\middle|\, x, y \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$.

   False. Since $\begin{bmatrix}x\\y\\1\end{bmatrix} \neq \mathbf{0}$ for all x, y, we have that $\mathbf{0} \notin W$. Hence W is not a subspace.

   (f) Let A be a square matrix and $c \in \mathbb{R}$. Then $\det(cA) = c \det A$.

   False. Let $A = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$. Then $\det(2A) = \begin{vmatrix}2 & 0\\0 & 2\end{vmatrix} = 4$, while $2 \det A = 2$.

(g) Suppose a 4×4 matrix A has the following two eigenspaces

$$E_2(A) = \text{Span}\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}\right\} \quad \text{and} \quad E_3(A) = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

Then A is not symmetric.

True. By the Spectral Theorem, every symmetric matrix is orthogonally diagonalizable. This implies that the eigenspaces of a symmetric matrix are necessarily orthogonal. Since $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \neq 0$, the eigenspaces $E_2(A)$ and $E_3(A)$ are not orthogonal. Hence A is not symmetric.

2. The matrix $A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & -1 & 3 & 2 \\ 1 & 2 & 1 & 3 \end{bmatrix}$ has RREF $R = \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 6 \end{bmatrix}$

(a) Are the columns of A linearly independent? Explain your reasoning.

No. Since A does not have a pivot in every column (since R doesn't), the columns of A are not linearly independent.

(b) Do the columns of A span $\mathbb{R}^3$? Explain your reasoning.

Yes. Since A has a pivot in every row (since R does), the columns span $\mathbb{R}^3$.

(c) What are the dimensions of the four fundamental subspaces? Explain your reasoning.

Since A has three pivots, $\text{rank}A = 3$. Hence $\dim C(A) = 3$ and $\dim C(A^T) = 3$. Since A is 3×4, we have that $\dim N(A) = 4 - 3 = 1$ and $\dim N(A^T) = 3 - 3 = 0$.

(d) Which of the four fundamental subspaces are subspaces of $\mathbb{R}^3$ and which are subspaces of $\mathbb{R}^4$? Explain your reasoning using the definitions of the four fundamental subspaces.

- Since $C(A)$ is the span of the columns of A, which are vectors in $\mathbb{R}^3$, $C(A)$ is a subspace of $\mathbb{R}^3$.
- Since $N(A)$ is the set of solutions to $A\mathbf{x} = \mathbf{0}$, where $\mathbf{x} \in \mathbb{R}^4$, $N(A)$ is a subspace of $\mathbb{R}^4$.
- Since $C(A^T)$ is the span of the columns of A^T, which are vectors in $\mathbb{R}^4$, $C(A^T)$ is a subspace of $\mathbb{R}^4$.
- Since $N(A^T)$ is the set of solutions to $A^T\mathbf{x} = \mathbf{0}$, where $\mathbf{x} \in \mathbb{R}^3$, $N(A^T)$ is a subspace of $\mathbb{R}^3$.

(e) Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation with associated matrix A.

Compute $T\left(\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}\right)$.

$$T\left(\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & -1 & 3 & 2 \\ 1 & 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$$

(f) Are there any nonzero vectors $\mathbf{v} \in \mathbb{R}^4$ satisfying $T(\mathbf{v}) = \mathbf{0}$? Explain your reasoning.

$T(\mathbf{v}) = \mathbf{0}$ if and only if $A\mathbf{v} = \mathbf{0}$ if and only if $\mathbf{v} \in N(A)$. Since $\dim N(A) = 1$, there exist nonzero vectors $\mathbf{v} \in \mathbb{R}^4$ satisfying $T(\mathbf{v}) = \mathbf{0}$.

3. Consider the following LU factorization of a matrix A.

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) Use this factorization to solve the system $A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (without row reduction). Explain your work.

We first solve $L\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ via forward substitution to get $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$. We next solve $U\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ via backward substitution to get $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$.

(b) Compute $\det A$. Show your work/explain your reasoning.

Since L and U are triangular, their determinants are the products of the diagonal entries. Hence $\det A = \det L \det U = (1)(1)(1)(-1)(2)(1) = -2$.

(c) Let T denote the linear transformation with associated matrix L and let S denote the linear transformation with associated matrix U. What is the linear transformation with associated matrix A? Be sure your notation is clear.

Since matrix multiplication corresponds to function composition, the linear transformation associated to $A = LU$ is $T \circ S$.

(d) Let $C \subset \mathbb{R}^3$ be the unit cube of volume 1. What is the volume of the image of C under the linear transformation given by A? Explain your reasoning.

$|\det A| = |-2| = 2$.

4. Consider the following QR factorization of a matrix A.

$$A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix} R$$

(a) Determine an orthonormal basis of $C(A)$. Explain your reasoning.

Since the QR factorization encode the Gram-Schmidt process, which provides an orthonormal basis of $C(A)$, the columns of Q form an orthonormal basis of $C(A)$.

(b) Determine the projection of $\mathbf{b} = \begin{bmatrix} 6 \\ 4 \\ 2 \\ 0 \end{bmatrix}$ onto $C(A)$. Show your work.

Since the columns of $Q = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix}$ are orthogonal, we can compute the projection as follows.

$$\begin{aligned} \text{proj}_{C(A)}\mathbf{b} &= \text{proj}_{\mathbf{q}_1}\mathbf{b} + \text{proj}_{\mathbf{q}_2}\mathbf{b} \\ &= (\mathbf{q}_1 \cdot \mathbf{b})\mathbf{q}_1 + (\mathbf{q}_2 \cdot \mathbf{b})\mathbf{q}_2 \\ &= 6\mathbf{q}_1 + 4\mathbf{q}_2 \\ &= \begin{bmatrix} 5 \\ 5 \\ 1 \\ 1 \end{bmatrix} \end{aligned}$$

(c) Does $A\mathbf{x} = \mathbf{b}$ have a solution (where $\mathbf{b}$ is the vector in part (b))? Explain your reasoning.

No. Since $\mathbf{b} \neq \text{proj}_{C(A)}\mathbf{b}$, we have that $\mathbf{b} \notin C(A)$ and so $A\mathbf{x} = \mathbf{b}$ does not have a solution.

5. Consider the following diagonalization of a matrix A.

$$A = PDP^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

(a) What are the eigenvalues of A? Explain your reasoning.

Since $A = PDP^{-1}$ is a diagonalization, the eigenvalues are the diagonal entries of D, namely $1, 2, 3$.

(b) Are the columns of A linearly independent? Explain your reasoning.

Yes. Since A does not have an eigenvalue of 0, A is invertible and so its columns are linearly independent.

(c) Write down a formula for A^k as a single matrix. Show your work.

We must first determine P^{-1}. We can do this with Gauss-Jordan elimination.

$$\left[P \mid I\right] = \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{Replace R2 with R2-2R3}} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{Replace R1 with R1+R2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -2 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Hence $P^{-1} = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$. Thus

$$\begin{aligned} A^k = (PDP^{-1})^k &= PD^kP^{-1} \\ &= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1-2^k & 2^{k+1}-2 \\ 0 & 2^k & 2(3^k-2^k) \\ 0 & 0 & 3^k \end{bmatrix} \end{aligned}$$

(d) Find a basis $\mathcal{B}$ of $\mathbb{R}^3$ built out of eigenvectors of A. Explain your reasoning.

The columns of P are three linearly independent eigenvectors of A; hence they form a basis of $\mathbb{R}^3$.

(e) Find the $\mathcal{B}-$coordinate vector of $\mathbf{v} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$. Explain your reasoning.

Since $\mathbf{v} = 0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, we have that $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$.

6. Consider the following SVD of a matrix A.

$$U\Sigma V^T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & .1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & -1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & -1/2 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$$

(a) Find a basis for $N(A)$. Explain your reasoning.

Since there are three singular values and rank$A = 3$. Hence the last two columns of V form a basis for $N(A)$.

(b) Determine the nonzero eigenvalues of A^TA. Explain your reasoning.

The nonzero eigenvalues of A^TA are the squares of the singular values of A; hence they are $(2\sqrt{2})^2 = 8$, $\sqrt{2}^2 = 2$, and $(.1)^2 = .01$.

(c) Does A^TA have a zero eigenvalue? If so, explain why and determine its algebraic multiplicity. If not, explain why.

Yes. Since A^TA is a 5×5 symmetric matrix, it has five real eigenvalues. Three of them are the nonzero eigenvalues found above. Hence the last two are 0. So, 0 is an eigenvalue with algebraic multiplicity 2.

(d) Find the SVD rank 2 approximation A_2 of A. Show your work.

$$\begin{aligned} A_2 &= \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T \\ &= 2\sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 & 0 \end{bmatrix} + \sqrt{2} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -1/2 & -1/2 & 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & -1/2 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1/2 & 1/2 & 3/2 & 3/2 & 0 \\ 3/2 & 3/2 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

(e) Do you expect the rank 2 approximation calculated above to be a good approximation of A? Explain your reasoning.

Yes. Since A has rank 3, $A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \sigma_3 \mathbf{u}_3 \mathbf{v}_3^T$. Since the last singular value $\sigma_1 = .1$ is close to zero, the first two terms of this sum (which is A_2) should give a good approximation of A.

Determine bases of $\mathbb{R}^5$ and $\mathbb{R}^4$ under which T is given by Σ.

7. Let $M = \begin{bmatrix} 0 & .5 & .5 \\ .5 & 0 & .5 \\ .5 & .5 & 0 \end{bmatrix}$

(a) Compute $\det M$. Show your work.

$$\begin{aligned} \begin{vmatrix} 0 & .5 & .5 \\ .5 & 0 & .5 \\ .5 & .5 & 0 \end{vmatrix} &= -.5 \begin{vmatrix} .5 & .5 \\ .5 & 0 \end{vmatrix} + .5 \begin{vmatrix} .5 & 0 \\ .5 & .5 \end{vmatrix} \\ &= -.5(-.25) + .5(.25) \\ &= .25 \end{aligned}$$

(b) Does M have complex eigenvalues? Explain your reasoning.

No, since M is symmetric, it's eigenvalues are all real by the Spectral Theorem.

(c) Explain why M must have an eigenvalue of 1.

Since the entries of M are nonnegative and each column sums to 1, M is a Markov matrix. Hence it must have an eigenvalue of 1.

(d) Find a basis for $E_1(M)$. Show your work.

Since $E_1(M) = N(M-I)$, we must determine a basis for $N(M-I)$. We first reduce $M-I$ to RREF:

$$M - I = \begin{bmatrix} -1 & .5 & .5 \\ .5 & -1 & .5 \\ .5 & .5 & -1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is no pivot in the third column, x_3 is the free variable. Hence the general solution to $(M-I)\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Hence $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for $E_1(M)$.

(e) Let $\mathbf{x}_0$ be any initial probability vector. Is it guaranteed that $M^k\mathbf{x}_0$ converges? If so, what explicit vector does it converge to? If not, explain why.

Since $M^2 = \begin{bmatrix} .5 & .25 & .25 \\ .25 & .5 & .25 \\ .25 & .25 & .5 \end{bmatrix}$ is a positive matrix, M is regular. Hence $M^k\mathbf{x}_0$ converges to the unique steady state vector $\mathbf{q}$. This vector is the unique probability vector in $E_1(M)$. Hence it is of the form $\begin{bmatrix} c \\ c \\ c \end{bmatrix}$, where $c + c + c = 1$. That is $\mathbf{q} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$.

8. Let A be an invertible matrix. Show that $|\det A|$ is the product of the singular values of A.

Let $A = U\Sigma V^T$ be an SVD. Then $|\det A| = |\det(U\Sigma V^T)| = |\det U||\det \Sigma||\det(V^T)|$. Since U and V^T are orthogonal, $|\det U| = |\det V^T| = 1$ and so $|\det A| = |\det \Sigma|$. Since Σ is a diagonal matrix whose diagonal entries are the singular values of A, we have that $|\det A|$ is the product of the singular values of A.

9. Prove the following part of the Diagonalization Theorem. If $A = PDP^{-1}$, where

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix},$$

then for each $1 \leq i \leq n$, $\mathbf{v}_i$ is an eigenvector of A with corresponding eigenvalue λ_i.

Multiplying both sides of $A = PDP^{-1}$ by P (on the right) gives $AP = DP$.

Now

$$AP = A\begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} A\mathbf{v}_1 & \cdots & A\mathbf{v}_n \end{bmatrix}$$

and

$$PD = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{v}_1 & \cdots & \lambda_n\mathbf{v}_n \end{bmatrix}$$

Since $AP = PD$, the ith column of AP equals the ith column of PD. That is, $A\mathbf{v}_i = \lambda_i\mathbf{v}_i$.